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Note

Solution of the Onsager Equation in Doubly Connected Regions

INTRODUCTION

A number of recent papers have independently derived a sixth order partial differential equation, here referred to as the Onsager equation, for the countercurrent flow in a gas centrifuge [1–4]. The Onsager equation is derived from the Navier–Stokes equations after linearization and the dropping of a number of terms and has been used in the above references to analyze the countercurrent gas flow in centrifuges. A number of articles on the centrifuge enrichment process have also noted and discussed gas flows in and around internal baffles [5–7], although the above derivations of the Onsager approximate equation have not, to our knowledge, treated the mathematical problem of how to solve the equation in a doubly connected domain such as would be created by a baffle.

A similar situation arises in solving the standard Poisson equation for the stream function in nonrotating incompressible liquid flow. Since the net flow around a body is fixed by the stream function value on the boundary of the object the problem is that of finding and stating in a useful form a constraint determining the boundary stream function. Sood and Elrod [8] were able to solve the nonrotating incompressible liquid case by requiring that the integral of the pressure gradient along arbitrary paths enclosing the body vanish. More recently a similar constraint was used by Israeli and Ungarish [9] to solve for the flow around a baffle in a liquid centrifuge.

In the following it is shown how the pressure continuity requirement can be applied within the context of the Onsager equation to solve for the unknown baffle stream function value. An explicit expression for the baffle stream function is obtained by choosing suitable paths enclosing the baffle, using boundary layer solutions so as to yield a formula involving integrals of the exterior stream function along the edges of the baffle. This expression can then be substituted for the boundary stream function yielding a well defined, although somewhat complicated, but explicit and consistent boundary condition at the baffle surface. Alternatively the expression for the baffle stream function can be used in a two step corrector method to find a solution consistent with the baffle stream function equation. This latter method is convenient for numerical work since it avoids the need to explicitly incorporate the boundary integrals. In this method a solution for an input guess at the baffle stream function is

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obtained after which the formula is used to correct the guess to the consistent baffle stream function value. The exact solution can be obtained after one correction since the equations are linear.

Equation for the Interior Boundary Stream Function

The geometry and notation are the same as used in the derivation of the Onsager equation given in [4]. For convenience the Onsager equation for the stream function ψ is restated here,

$$(e^x(e^x\psi_x)_{xx})_{xxx} + \frac{\mathrm{Re}^2 S}{16A^{12}}\psi_{yy} = F_x(x, y) \tag{1}$$

where x and y are the nondimensional radial and axial coordinates, and where $F_x(x, y)$ represents nonhomogeneous forcing terms arising from external sources of mass, momentum, and energy.

Assuming a baffle having some axial thickness, one can derive a formula for the baffle stream function using the paths enclosing the baffle shown in Fig. 1. Along the sections $x = \text{constant}$ the axial pressure gradient can be gotten from the axial momentum equation yielding,

$$(e^x p)_y = \frac{8A^6}{\mathrm{Re}} e^x W_{xx} = -\frac{16A^8}{\mathrm{Re}} e^x(e^x\psi_x)_{xx}, \tag{2}$$

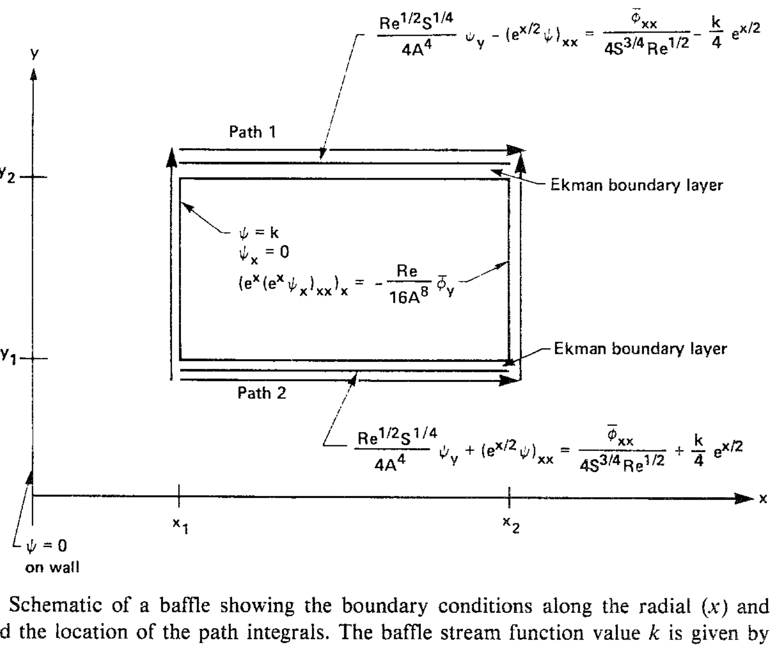


FIG. 1. Schematic of a baffle showing the boundary conditions along the radial (x) and axial (y) surfaces and the location of the path integrals. The baffle stream function value k is given by Eq. (9).

where ψ is the stream function. Along the sections $y=$ constant the radial pressure gradient is obtained from the radial momentum equation,

$$(e^x p)_x = \phi. \tag{3}$$

Substituting for ϕ using the Ekman boundary conditions gives, along the upper surface, $y = y_2$,

$$(e^x p)_x = \bar{\phi}(x, y_2) + 4S^{3/4}\mathrm{Re}^{1/2} e^{x/2}(\psi(x, y_2) - k) \tag{4}$$

and along the lower surface, $y = y_1$,

$$(e^x p)_x = \bar{\phi}(x, y_1) - 4S^{3/4}\,\mathrm{Re}^{1/2} e^{x/2}(\psi(x, y_1) - k) \tag{5}$$

where k is the unknown stream function value of the baffle, equal to the net mass flow circulating around the baffle, and $\psi(x, y_2)$ and $\psi(x, y_1)$ are the interior stream function values above and below the baffle.

Integrating the pressure gradient along the two paths enclosing the baffle yields, for path 1 along the upper side,

$$\begin{aligned} e^{x_2}p(x_2, y_2) = {} & e^{x_1}p(x_1, y_1) + \int_{y_1}^{y_2} (e^x p)_y \,|_{x=x_1}\, dy \\ & + \int_{x_1}^{x_2} \bar{\phi}(x, y_2)\, dx + 4S^{3/4}\,\mathrm{Re}^{1/2} \int_{x_1}^{x_2} e^{x/2}(\psi(x, y_2) - k)\, dx \end{aligned} \tag{6}$$

and for path 2 along the lower side,

$$\begin{aligned} e^{x_2}p(x_2, y_2) = {} & e^{x_1}p(x_1, y_1) + \int_{y_1}^{y_2} (e^x p)_y \,|_{x=x_2}\, dy \\ & + \int_{x_1}^{x_2} \bar{\phi}(x, y_1)\, dx - 4S^{3/4}\,\mathrm{Re}^{1/2} \int_{x_1}^{x_2} e^{x/2}(\psi(x, y_1) - k)\, dx. \end{aligned} \tag{7}$$

Subtracting Eq. (7) from Eq. (6) and noting that continuity of pressure around the baffle resuires the same result independent of path one gets,

$$\begin{aligned} 0 = {} & \int_{y_1}^{y_2} [e^x p_y \,|_{x=x_1} - e^x p_y \,|_{x=x_2}]\, dy \\ & + 4S^{3/4}\,\mathrm{Re}^{1/2} \int_{x_1}^{x_2} e^{x/2}[\psi(x, y_2) + \psi(x, y_1) - 2k]\, dx \\ & + \int_{x_1}^{x_2} [\bar{\phi}(x, y_2) - \bar{\phi}(x, y_1)]\, dx. \end{aligned} \tag{8}$$

Rearranging terms, substituting Eq. (2), and solving for k then yields,

$$
\begin{aligned}
k = {} & 4S^{3/4}\,\mathrm{Re}^{1/2}\int_{x_1}^{x_2}[\psi(x, y_2) + \psi(x, y_1)]\,dx \\
& - \frac{16A^8}{\mathrm{Re}}\int_{y_1}^{y_2}[e^x(e^x\psi_x)_{xx}\,|_{x=x_1} - e^x(e^x\psi_x)_{xx}\,|_{x=x_2}]\,dy \\
& + \int_{x_1}^{x_2}[\bar{\phi}(x, y_2) - \bar{\phi}(x, y_1)]\,dx \\
& \div 16S^{3/4}\,\mathrm{Re}^{1/2}(e^{x_2/2} - e^{x_1/2}).
\end{aligned} \tag{9}
$$

Equation (9) gives the net mass circulation around the baffle in terms of the interior stream function and its derivatives along the boundary and in terms of the temperature boundary conditions along the surface of the baffle specified by

$$\bar{\phi}(x, y_1) = T(x, y_1)/T_0 \tag{10}$$

$$\bar{\phi}(x, y_2) = T(x, y_2)/T_0. \tag{11}$$

Discretization of the Onsager equation, see, for example [10], leads to a set of simultaneous linear equations for the stream function on a set of grid points which is easily solved by standard methods. Boundary condition (9) can be similarly discretized and included explicitly in the matrix of coefficients but this is algebraically complicated and so susceptible to error.

An alternative method requires twice the computer time but is easier to implement. Since k in Eq. (9) is linear in ψ one can solve for k in two steps. First, solve the Onsager equation with homogeneous boundary and forcing conditions except for a fixed input non zero stream function value k on the baffle, call it k_0. After obtaining the solution then calculate a value of k from Eq. (9) using this solution, call it k_1. Next compute the solution to the Onsager equation for the desired nonhomogeneous boundary and forcing conditions, but with the stream function k on the baffle identically zero. After obtaining this solution use it to calculate a k from Eq. (9) call it k_2. The solution for k consistent with the desired nonhomogeneous boundary and forcing conditions is then given by

$$k = \frac{k_0 k_2}{k_0 - k_1}. \tag{12}$$

Equation (12) results because k computed from Eq. (9) is linear in the input value k_0 and so can be solved for k [by Eq. (9)] $= k$ [fixed input] in two steps. The penalty of doubled computational time is negligible as a single solution of the Onsager equation requires little computer time.

The pressure continuity condition implies some radial flows into and out of the Ekman layers around the edges at x_1 and x_2. In a numerical solution these flows are included automatically if normal boundary conditions along the edges between y_1 and y_2 are included. These boundary conditions, along the edges $x = x_1$ and $x = x_2$, are

$$\psi = k \tag{13}$$

$$\psi_x = 0 \tag{14}$$

$$(e^x(e^x\psi_x)_{xx})_x = -\frac{\mathrm{Re}}{16A^8}\bar{\phi}_y. \tag{15}$$

Equation (13) is necessary because the stream function at the baffle edge must equal the net mass circulation around the baffle. Equation (14) is the no slip boundary conditions on the axial velocity component, and Eq. (15) gives the temperature boundary condition identical to that at the rotor wall where $\psi = 0$. Figure 1 shows these boundary conditions as they apply along the baffle surfaces.

Examples from a test computation involving a doubly connected region are shown in Figs. 2(a) and (b). The plots show the axial gas velocity inside a short bowl rotating about a vertical axis with a 1% temperature perturbation on the lower end plate at $y = 0$ given by,

$$\Delta T/T_0 = 0.01(1 - e^{-x}). \tag{16}$$

The two examples show the effect of inserting a thin disk (thickness 1.25% of the axial length) at $y = \frac{1}{2}y_0$ which rotates at the same angular velocity as the bowl. The outer and inner edges of the disk are located at radii corresponding to $x = 1$ and 2.

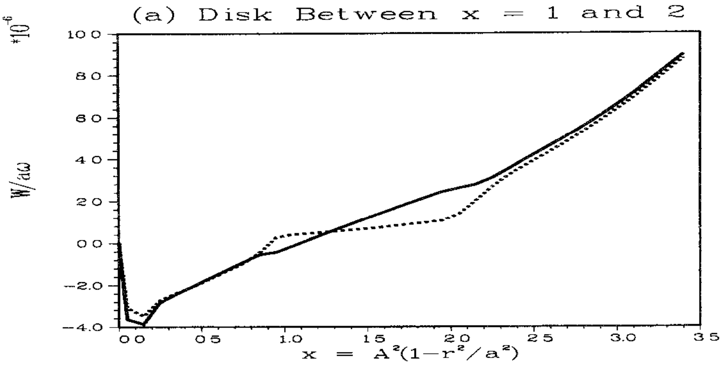
The bowl, which has diameter and axial length 18.29 cm and 4.57 cm respectively, rotates at 348 hz ($A^2 = 10$). The gas which is at 1 atm pressure at the outer wall, is heated in the Ekman boundary layer on the lower end plate, becomes buoyant and flows through the boundary layer toward the axis. Ekman pumping into and out of the end plate boundary layer produces the axial countercurrent flow shown in Figs. 2(a) and (b). At this pressure and rotation rate the behavior of the flow is very similar to that in the incompressible case discussed in [9], i.e., the interior flow depends only on the radius.

The case without the disk can be solved analytically. The solution,

$$W/a\omega = (\Delta T/T_0)A^2S^{-3/4}\,\mathrm{Re}^{-1/2}[(1/8)\,e^{x/2} - (3/8)\,e^{-x/2}] \tag{17}$$

is plotted as the solid line in Fig. 2(b). The numerical solution at axial positions $y = \frac{1}{4}y_0$ (squares) and $\frac{3}{4}y_0$ (circles) is plotted at the mesh points. The agreement between the numerical and analytical solutions is good; the small differences at the right boundary are due to the numerical treatment of the boundary condition at large x. There is also a vertical boundary layer along the rotor wall at $x = 0$ such that $W = 0$ on the boundary.

The case with the disk inserted is shown in Fig. 2(a). The axial velocity plots at $y = \frac{1}{4}y_0$ (solid line) and at $y = \frac{3}{4}y_0$ (dashed line) show that a Taylor column is generated by the disk. In addition to the narrow shear layer along the wall at $x = 0$,



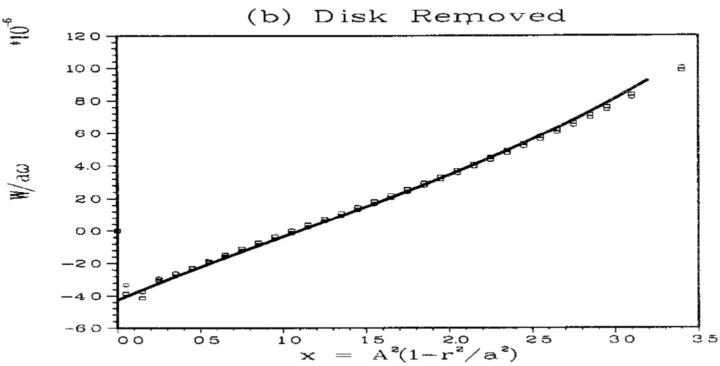


FIG. 2. Generation of a Taylor column in a short bowl by a thin disk with edges at $x = 1$ and 2. (a) Normalized axial velocity at $y = \frac{1}{4} y_0$ (solid curve) and $y = \frac{3}{4} y_0$ (dashed curve) from a numerical solution of the Onsager equation using the methods described here and in [10]. (b) Comparison between the numerical solution at $y = \frac{1}{4} y_0$ and $\frac{3}{4} y_0$ (squares and circles) and the analytical solution (solid curve) when the disk is removed.

shear layers form along the boundaries of the Taylor column at $x = 1$ and 2. Although I could not solve the disk case by analytical methods because there is a small variation with y the calculation can be checked by computing the pressure difference between two points in the flow using integration paths passing on opposite sides of the baffle. The pressures computed by alternative paths agree to at least six digits.

CONCLUSION

The pressure continuity constraint for flow around a baffle can be quite easily applied to the Onsager equation. It can be used to derive a formula for the circulation around a baffle with some axial thickness, which can then be used in a two step solution method. The formula could also be incorporated explicitly into the matrix of coefficients resulting from discretization of the Onsager equation but this approach would require considerably more algebra. The first method is recommended. The radial flows into and out of the Ekman layers are included naturally when provision for axial thickness of the baffle in included. Boundary conditions along surfaces parallel to the rotor wall are similar to those at the rotor wall and can be treated by the same numerical methods.

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